Exercises to *1.8 Natural deduction*

(1) *Use natural deduction to prove the following inferences:*

1)
1. $A \to C$
2. $D \lor (A \land B)$
3. $\neg D \quad / \; C$

2)
1. $I \to J$
2. $J \lor (I \lor K)$
3. $J \to M$
4. $\neg M$
5. $K \to \neg N \quad / \; \neg N$

3)
1. $A \to C$
2. $C \to (A \leftrightarrow B)$
3. $\neg B$
4. $\neg D \to [D \lor \neg (A \leftrightarrow B)]$
5. $D \to B \quad / \; \neg A \land \neg B$

4)
1. $A \to B$
2. $(A \land B) \to (\neg\neg C \land \neg D)$
3. $\neg E \to \neg C$
4. A
5. $\neg\neg E \to F \quad / \; F \lor (\neg A \leftrightarrow \neg B)$

5)
1. $F \to G$
2. $\neg (F \land G) \land L$
3. $(G \lor H) \to (I \land J)$
4. $F \lor (K \lor G)$
5. $\neg K \land L \quad / \; I \lor H$

6)
1. $(A \lor B) \land C$
2. $A \to D$
3. $B \to E$
4. $E \to F \quad / \; D \lor F$

7)
1. $Q \to (V \lor R)$
2. $T \to \neg U$
3. $\neg V$
4. $T \lor Q$
5. $\neg U \to V \quad / \; R$

8)
1. $\neg S \to (N \lor O)$
2. $(N \to U) \land (P \to T)$
3. $(O \to T) \land (P \to N)$
4. $\neg U$
5. $S \to U \quad / \; T$

9)
1. $A \to B$
2. $\neg B$
3. $[(\neg A \land \neg B) \lor C] \to (B \lor D) \; / \; D \lor \neg E$

10)
1. $(Q \lor R) \to [(S \lor L) \to \neg T]$
2. $(S \lor U) \to Q$
3. $S \land \neg U$
4. $T \lor K \quad / \; K$

(2) For each step in the following natural deduction proofs, determine from which previous lines and by which inference or equivalence schemes are obtained:

1)
1. $A \lor \neg B$
2. $\neg C \to \neg A \quad / \; B \to C$

6)
1. $[O \to (P \land Q)] \land [R \to (P \land S)]$
2. $[(T \to \neg O) \land U] \to X$

3. $\neg B \lor A$
4. $B \to A$
5. $A \to C$
6. $B \to C$

2)
1. $P \to \neg P$
2. $(P \land Q) \lor (R \land S)$ $\quad /R$
3. $\neg P \lor \neg P$
4. $\neg P$
5. $\neg P \lor \neg Q$
6. $\neg(P \land Q)$
7. $R \land S$
8. R

3)
1. $A \to (B \to C)$
2. $(\neg C \lor \neg D) \lor F$
3. $\neg E \to (D \land \neg F)$ $\quad / A \to (B \to E)$
4. $(A \land B) \to C$
5. $\neg(C \land D) \lor F$
6. $(C \land D) \to F$
7. $C \to (D \to F)$
8. $(A \land B) \to (D \to F)$
9. $\neg(D \land \neg F) \to \neg\neg E$
10. $\neg(D \land \neg F) \to E$
11. $(\neg D \lor \neg\neg F) \to E$
12. $(\neg D \lor F) \to E$
13. $(D \to F) \to E$
14. $(A \land B) \to E$
15. $A \to (B \to E)$

4)
1. $K \to L$
2. $M \to L$
3. $N \to [K \lor (K \lor M)]$
4. N $\quad /L$

3. $(U \to X) \to (T \land R)$ $\quad / Q \lor S$
4. $(T \to \neg O) \to (U \to X)$
5. $(T \to \neg O) \to (T \land R)$
6. $\neg(T \to \neg O) \lor (T \land R)$
7. $\neg(\neg T \lor \neg O) \lor (T \land R)$
8. $(\neg\neg T \land \neg\neg O) \lor (T \land R)$
9. $(T \land O) \lor (T \land R)$
10. $T \land (O \lor R)$
11. $(O \lor R) \land T$
12. $O \lor R$
13. $(P \land Q) \lor (P \land S)$
14. $P \land (Q \lor S)$
15. $(Q \lor S) \land P$
16. $Q \lor S$

7)
1. $P \leftrightarrow \neg Q$
2. $(P \to S) \land (S \to P)$
3. $S \to Q$ $\quad / Q$
4. $(P \to \neg Q) \land (\neg Q \to P)$
5. $P \to \neg Q$
6. $\neg\neg Q \to \neg P$
7. $Q \to \neg P$
8. $S \to \neg P$
9. $P \to S$
10. $P \to \neg P$
11. $\neg P \lor \neg P$
12. $\neg P$
13. $(\neg Q \to P) \land (P \to \neg Q)$
14. $\neg Q \to P$
15. $\neg\neg Q$
16. Q

8)
1. $A \to (C \lor \neg B)$
2. $(B \land C) \to (A \land D)$

5. $K \vee (K \vee M)$
6. $(K \vee K) \vee M$
7. $K \vee M$
8. $(K \to L) \wedge (M \to L)$
9. $L \vee L$
10. L

5)
1. $P \to R$
2. $(T \to \neg S) \to \neg R$ / $P \to T$
3. $\neg\neg R \to \neg(T \to \neg S)$
4. $R \to \neg(T \to \neg S)$
5. $\neg R \vee \neg(T \to \neg S)$
6. $\neg R \vee \neg(\neg T \vee \neg S)$
7. $\neg R \vee (\neg\neg T \wedge \neg\neg S)$
8. $\neg R \vee (T \wedge \neg\neg S)$
9. $\neg R \vee (T \wedge S)$
10. $(\neg R \vee T) \wedge (\neg R \vee S)$
11. $\neg R \vee T$
12. $R \to T$
13. $P \to T$

3. B / $A \leftrightarrow C$
4. $A \to (\neg B \vee C)$
5. $A \to (B \to C)$
6. $(A \wedge B) \to C$
7. $(B \wedge A) \to C$
8. $B \to (A \to C)$
9. $A \to C$
10. $B \to [C \to (A \wedge D)]$
11. $C \to (A \wedge D)$
12. $\neg C \vee (A \wedge D)$
13. $(\neg C \vee A) \wedge (\neg C \vee D)$
14. $\neg C \vee A$
15. $C \to A$
16. $(A \to C) \wedge (C \to A)$
17. $A \leftrightarrow C$

(3) *Use natural deduction to prove the following inferences:*

1)
1. $W \to M$
2. $\neg W \to E$ / $M \vee E$

2)
1. $(R \to S) \wedge (P \to Q)$
2. $(S \wedge Q) \to O$
3. $\neg O$ / $\neg R \vee \neg P$

3)
1. $(A \vee B) \to (C \wedge D)$
2. $C \to E$
3. $\neg E$ / $\neg A$

4)
1. $K \to L$
2. $K \to M$ / $(\neg L \vee \neg M) \to \neg K$

6)
1. $S \to L$ / $S \to (L \vee W)$

7)
1. $A \leftrightarrow B$
2. $C \to \neg B$ / $A \to \neg C$

8)
1. $E \to (F \wedge G)$
2. $(F \vee H) \to I$ / $E \to I$

9)
1. $(A \vee B) \to (C \wedge D)$ / $\neg A \vee C$

5) 1. $(L \land M) \to N$
2. $(L \land \neg M) \to \neg N$ / $L \to (M \leftrightarrow N)$

10) 1. $M \lor (N \land O)$
2. $M \to O$ / O

(4) *Solve the examples from the previous exercise this time using reductio ad absurdum or conditional proof.*

(5) *Prove the following arguments by natural deduction using the letters given in parentheses to symbolize the simple statements in them. Try to prove them in two ways – first without using indirect or conditional proof and then using one of them.*[1]

1) If I start a new job (J), I will have to buy a car (C), and if I go to the sea this summer (S), I will spend half of my savings (H). But if I buy a car and spend half of my savings, I will have to live on 10 euros a day (L). However, I cannot live on 10 euros a day. So either I won't buy a car or I won't go to the sea.

2) If our representative runs for president (P), if he runs a positive campaign (C), there will be a runoff (R). If there is a runoff and he wins the election (E), he will not be reelected in four years (Y). However, if he supports the death penalty (D), he will win the election and be reelected in four years. Therefore, if our representative runs for president, if he runs a positive campaign, he will not support the death penalty.

3) If the doctor injects the antibodies (I), the patient will have an allergic reaction (A), and if he has an allergic reaction, his kidney will stop functioning (K). But if the doctor does not inject the antibodies, the bacteria will spread to the bloodstream (B). The patient's kidney will stop functioning if the bacteria spreads to the bloodstream. If the kidney stops functioning, the patient will not survive until the morning (S). Therefore, the patient will certainly not survive until the morning.

4) On New Year's Eve (N), John drinks red wine (W). If he celebrates with friends (F), he drinks beer (B). Therefore, if John celebrates the New Year with friends, he drinks red wine and beer.

5) If Alice enrolls in Ancient Greek (G), she will also enroll in Latin (L). If she enrolls in Ancient Greek and Latin, she will enroll in logic (O). But if she enrolls in Ancient Greek, then if she enrolls in logic, she will enroll in mathematics (M). Therefore, if Alice enrolls in Ancient Greek, she will also enroll in mathematics.

6) If an economic crisis begins (E) or international conflicts arise (C), if the government fails to act (F) or takes inadequate action (I), there will be neither economic growth (G) nor political stability (S). If there is no economic growth or taxes are raised (T), there will be protests (P). Therefore, if an economic crisis begins, then if the government fails to act, there will be protests.

[1] Most of the examples are from exercise **(2)** in **1.7 Quick test of logical inference.** There, the validity of the arguments had to be proven by a quick test. You can compare the two methods.

7) If Peter has met Mary (M), he would have told her the news (T) if he knew it (K). But Peter met Mary and did not tell her the news. So he didn't know it.

8) If I start my own business (B), I will become rich (R), and if I start a scientific career (S), I will have free time (T). I will either start my own business or start a scientific career. However, if I start my own business, I will not have free time, whereas if I start a scientific career, I will not be rich. Therefore, I will have free time if and only if I am not rich.

9) If I go to the ball (B), I'll have to buy a tailcoat (T). But if I buy a tailcoat, I will not be able to pay my rent (R) and repay my loan (L) at the same time. If I don't pay the rent, I'll have to hide from the landlord (H), and I can't do that. Besides, I'll have to repay the loan. So, I can't go to the ball.

10) If taxes increase (T), unemployment will increase (U), and if investments decrease (I), growth (G) will decrease. If unemployment increases or investments decrease, consumption will fall (C). Consumption has not decreased. Therefore, neither taxes have been increased nor investments have been reduced.

11) If John is drinking with friends (F), tomorrow he will have a hangover (H). If his favorite team has lost (L) again, tomorrow he will be irritable (I). Therefore, if John is drinking with friends or his favorite team has lost, tomorrow he will have hangover or be irritable.

12) If Jupiter has been under the influence of Mars (M) at the beginning of the year, there will be war (W) or civil unrest (U) during the year. If Jupiter has been under the influence of Saturn (S) at the beginning of the year, either the year will be hungry (H) or there will be war. However, there will certainly be no war. Therefore, if the year will be neither hungry nor there will be civil unrest, Jupiter has been under the influence neither of Mars nor Saturn.

(6) *Use natural deduction to answer the following questions:*

1) Imagine that you are a detective and have the following information. There are four suspects – call them "P", "Q", "R" and "S". If P is innocent, then S is also innocent, but R's guilt will be certain. If S is innocent, then Q is among the perpetrators of the crime. If S is guilty, then so is R. R, however, has a reliable alibi. Who are guilty and who are innocent?

2) As in the previous exercise but this time the information is as follows. P is guilty if and only if Q is innocent. R is innocent if and only if S is guilty. If S is among the perpetrators, then P is also among them, and vice versa. If S is guilty, then so is Q.

3) In a restaurant, a customer tells the waiter the following: "I eat potatoes or rice, but not both at the same time. If I eat potatoes, then I don't eat bread. If I eat bread or don't eat potatoes, then I don't eat rice". What should the waiter serve?